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Approximate Solution for a Class of Optimal Control Problems in Distributed Parameter Systems

Kamil Mamtiyev[1], Ulviyya Rzayeva[2*], Hafiz Bayramov[3], Konul Mirzammadova[4]

[1,2,3,4]Department of Digital Technologies and Applied Informatics, UNEC, Baku, Azerbaijan*
[1]0000-0002-5842-6085, kamil.mamtiyev@unec.edu.az
[2]0000-0001-5881-6633, ulviyya.rzayeva@unec.edu.az*
[3]0000-0003-4588-9483, hafiz.bayramov@unec.edu.az
[4]0000-0001-6047-4724, konul.mirza@unec.edu.az

Abstract

An approximate solution is developed using the direct method for a specific class of optimal control problems in systems governed by nonlinear boundary value problems of parabolic type. These problems are particularly significant in the modeling and optimization of dynamic processes distributed over space and time. The methodology is based on constructing finite-dimensional approximations of the original infinite-dimensional problem, allowing for a practical computational approach. By applying a priori estimate for the solutions of systems of linear ordinary differential equations, the convergence of the proposed direct method is rigorously proven. This convergence guarantees that the approximate solutions approach the exact solution of the original control problem in terms of minimizing the given functional. Furthermore, a constructive scheme for generating a minimizing sequence of controls is introduced, which depends on the chosen class of admissible controls. This scheme provides a systematic way to approach optimality in practical applications. As a practical illustration, the study presents an example related to determining the optimal technological regime for the operation of gas wells, which demonstrates the applicability of the proposed method to real-world engineering problems. The developed approach can serve as a valuable computational tool for solving similar optimal control problems in distributed parameter systems.

Keywords: boundary value problems, direct method, convergence in terms of the functional, minimizing sequence

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1. Introduction

The foundations of theoretical research and practical developments in distributed parameter systems were laid more than half a century ago in the papers of Butkovsky [1]. Since then, the theory of control for distributed parameter systems has been enriched with new ideas and results. Each year, more publications emerge across various subfields. However, many significant aspects of this theory remain underdeveloped, particularly in the context of optimal control problems for systems with processes that are described by nonlinear boundary value problems of parabolic type. In addition to the complexity introduced by the nonlinearity of boundary problems, constraints on control actions and the system's state variables necessitate approximate optimization methods.

It is also worth noting that the relevance of studying optimal control problems for nonlinear distributed parameter systems stems from the fact that, in real-world systems, control is typically exercised via devices with lumped parameters. Optimal control problems of this type were first addressed in the mid-1960s by Yegorov, who primarily derived necessary optimality conditions in the form of the Pontryagin maximum principle. From the early 1970s onwards, as such problems gained prominence, they became the focus of numerous researchers [2–6].

In [2], the problem of suppressing oscillations in a coupled system was examined, where the system is described by a set of wave equations and a second-order ordinary differential equation. It is assumed that the control function and the lumped parameter object act on the left and right boundaries of the distributed system, respectively. The system's state functions are linked through the boundary conditions of the wave equation. The problem was solved using D'Alembert's formula, and finite-dimensional approximations were constructed using the direct method. A numerical solution to this problem was later obtained in [3], where the direct method was combined with a gradient projection method using a specially chosen step size. The results confirmed the convergence of the method for the functional. Problems of this nature can arise, for instance, in the control of gas flows in long pipelines or electromagnetic oscillations in extended electrical lines – processes that can be controlled using lumped parameter devices.

In [4], an optimal control problem governed by a nonlinear two-dimensional gas filtration equation was considered, though no numerical results were provided. A numerical solution to an optimal control problem governed by a two-dimensional gas filtration equation in a porous medium with a low-permeability overburden was obtained in [5]. The method of averaging was used to reduce the original boundary problem to a one-dimensional equation, and the direct method was employed for the numerical solution.

In [6], an optimal technological regime for gas well operation was determined by regulating bottom-hole pressure within a specified range. This problem reduces to an optimal control problem for a system governed by a nonlinear one-dimensional unsteady gas filtration equation in a porous medium and an ordinary differential equation.

In the present paper, the direct method is used to approximately solve a class of optimal control problems for processes described by nonlinear parabolic equations with initial and boundary conditions. For the problem under consideration, using known priori estimates for the solutions of systems of linear ordinary differential equations, the convergence of the solution of the approximating boundary value problem to the solution of the original problem is proven. Based on this, the convergence of the approximate solution of the corresponding optimal control problem in terms of the functional is also established. A constructive scheme for building optimal control is proposed.

2. Experiments

Let there be some controlled processes in the domain $Q = \{0 \le x \le 1, \quad 0 \le t \le T\}$that are described by a boundary value problem of the following type

$$u_t = F(x, t, u, u_x, u_{xx}, \alpha),\ F_{u_{xx}} \ge a_0 = const > 0, \tag{2.1}$$

$$u(0, t) = a(t),\ u(1, t) = b(t),\ 0 \le t \le T, \tag{2.2}$$

$$u(x, 0) = \omega(x),\ 0 \le x \le 1, \tag{2.3}$$

It is assumed that the function $F(x, t, p, q, r, \alpha)$ is continuous and sufficiently smooth for all $(x, t) \in Q$ and all real values p, q, r and α. It is also assumed that $F, F_p, F_{q_x}, F_r, F_\alpha$ are uniformly bounded for the specified values of its arguments and continuous in *t.* Functions $a(t), b(t)$ and $\omega(x)$ are continuous in the corresponding domains and satisfy the following agreement conditions $a(0) = \omega(0), b(0) = \omega(1)$. The control function $\alpha = \alpha(x, t)$ takes values from some closed domains, defined, for example, by the inequalities $|\alpha(x, t)| \le 1$ and in the domain Q, it has a finite number of non-intersecting smooth discontinuity lines. We will call such control *admissible*. The problem of finding the function $u(x,t)$ from conditions (2.1) – (2.3) with fixed control is called a *direct problem*.

It is required to find such an admissible control $\alpha = \overline{\alpha}(x,t)$ and the corresponding solution $\bar{u}(x,t)$ of the direct problem (2.1) – (2.3), so that the functional

$$S(\alpha) = \int_o^1 G\big(x, u(x,T)\big)dx, \tag{2.4}$$

takes the smallest possible value, where $G(x,u)$ is a given continuously differentiable function of its arguments, and the functions G_x, G_u are uniformly bounded. Functional (2.4) is chosen in this form only to shorten the notation. Note that the issues of solvability of the boundary value problem (2.1) – (2.4) by the method of lines are studied in [7].

The method of lines is used for an approximate solution of problem (2.1) – (2.4). Let $\overline{\omega}_h$ uniform grid of lines in a segment $0 \le x \le 1$ with the distance h between nodal points $x_h^i = ih, i = 0,1,\dots,N, Nh = 1$. Let us denote by $\phi_h^i(t)$ value of an arbitrary function $\phi(x,t)$ in nodes x_h^i of a grid $\overline{\omega}_h$, and in the internal nodes of this grid we replace the direct problem (2.1) – (2.3) with a system of differential-difference equations:

$$\frac{du_h^i}{dt} = F\left(x_h^i, t, u_h^i, \frac{\Delta_c u_h^i}{2h}, \frac{\Delta^2 u_h^i}{h^2}, \alpha_h^i(t)\right), \ 1 \le i \le N-1, \tag{2.5}$$

$$u_h^0 = a(t), \ u_h^N = b(t) \tag{2.6}$$

with initial conditions

$$u_h^i(0) = \omega\big(x_h^i\big), \quad i = 0,1,\dots,N, \tag{2.7}$$

Thus, problem (2.1) – (2.4) is reduced to the choice of function $\bar{\alpha}_h^i(t)$ from the conditions of the minimum functional

$$S_h\left(\alpha_h^i(t)\right) = h\sum_{i=0}^{N-1} G\left(x_h^i, u_h^i(T)\right) \tag{2.8}$$

under conditions (2.5) – (2.7).

Let us denote by $\delta_h^i(t) = u\big(x_h^i, t\big) - u_h^i(t), i = 0,1,\dots,N,$where $u\big(x_h^i, t\big)$ is the exact solution of problem (2.1) – (2.3), and $u_h^i(t)$ is solution of the differential-difference problem (2.5) – (2.7). Substituting into (2.5) – (2.7) the exact solution $u(x_h^i, t)$ of the direct problem (2.1) – (2.3), we have:

$$\frac{\partial u\big(x_h^i,t\big)}{\partial t} = F\left(x_h^i, t, u\big(x_h^i, t\big), \frac{\Delta_c u\big(x_h^i,t\big)}{2h}, \frac{\Delta^2 u\big(x_h^i,t\big)}{h^2}, \alpha_h^i(t)\right) + O(h^2), \ 1 \le i \le N-1, \tag{2.9}$$

$$u(x_h^0, t) = a(t), u(x_h^N, t) = b(t), \tag{2.10}$$

where

$\Delta_c u(x_h^i, t) = u(x_h^{i+1}, t) - u(x_h^{i-1}, t), \Delta^2 u(x_h^i, t) = u(x_h^{i+1}, t) - 2u(x_h^i, t) + u(x_h^{i-1}, t).$

Subtracting from (2.9) and (2.10), respectively, the relations (2.5) and (2.6), we compose for the errors $\delta_h^i(t)$ system of inhomogeneous equations:

$$\frac{d\delta_h^i(t)}{dt} = \tilde{F}_p \delta_h^i(t) + \tilde{F}_q \frac{\Delta_c \delta_h^i(t)}{2h} + \tilde{F}_r \frac{\Delta^2 \delta_h^i(t)}{h^2} + O(h^2), \ \ 1 \le i \le N-1, \tag{2.11}$$

$$\delta_h^i(t) = 0, \ \ \delta_h^N(t) = 0 \tag{2.12}$$

with zero initial data

$$\delta_h^i(0) = 0, \quad i = 0,1,\dots,N, \tag{2.13}$$

where by the icon ~ the values of the derivatives at intermediate points are indicated.

Applying the well-known estimate from [9] for solutions of a system of linear inhomogeneous ordinary differential equations to the solution $\delta_h^i(t)$ we have

$$\max_{1\le i\le N-1}\left|\delta_h^i(t)\right| \le \int_0^T O(h^2)\, e^{\int_s^t \max c(x_h^i,\tau)d\tau} ds, \tag{2.14}$$

where $c(x_h^i,t) = \tilde{F}_p$. From inequalities (2.14) it follows that the solution of the differential-difference boundary value problem (2.5) – (2.7) tends with the speed $O(h^2)$ under $h \to 0$ to the solution of the boundary value problem (2.1) – (2.3).

If we denote

$$S_h^0 = \inf_{\alpha_n^i} S_h(\alpha_h^i) = S_h\left(\bar{\alpha}_h^i(t)\right).$$

After applying the Cauchy-Bunyakovsky inequality, it can be shown that the inequality holds

$$\left|S(\alpha(x,t)) - S_h\left(\alpha_h^i(t)\right)\right| \le Ch, \tag{2.15}$$

where *C* does not depend on *h*. From (2.15) it follows that

$$\lim_{h\to 0} S_h^0 = \min_{\alpha} S(\alpha) = S\left(\bar{\alpha}(x,t)\right),$$

that is, there is convergence in functional.

THEOREM. Let $\bar{\alpha}_h^i(t)$ is the optimal control in the approximating optimal problem, and $\bar{\alpha}_h^i(x,t)$ is a continuation of function $\bar{\alpha}_h^i(t)$ from the grid $\bar{\omega}_h$ for all domain *Q*. Then the sequence of controls $\bar{\alpha}_h^i(x,t)$ is minimizing for the functional (2.4) in problem (2.1) – (2.4).

PROOF. Let h_m is a sequence of positive numbers that tends to zero under $m \to \infty$, and $\bar{\alpha}_{h_m}^i(t)$ is a sequence of optimal solutions of the approximating problem (2.5) – (2.8), corresponding to h_m. Let's continue the functions $\bar{\alpha}_{h_m}^i(t)$ from a grid of lines $\bar{\omega}_h$ for the whole domain *Q*, considering, in particular, for $x_h^i \le x \le x_h^{i+1}$, $0 \le t \le T$, $i = 0,1,\ldots,N-1$.

Let us prove that when $m \to \infty$ the control sequence $\bar{\alpha}_{h_m}^i(x,t)$ is minimizing for the functional (2.4). For definiteness, we assume that the functional (2.4) on the set of admissible controls has a finite lower bound. Let $\alpha_m(x,t)$ is some other minimizing sequence for the functional (2.4), that is,

$$\lim_{m\to\infty} S(\alpha_m(x,t)) = \inf_{\alpha} S(\alpha) < +\infty \tag{2.16}$$

In the approximating optimal problem (3.1) – (3.3), instead of $\alpha_h^i(t)$ we put $\alpha_m\left(x_{h_m}^i,t\right)$. Then considering that $\bar{\alpha}_{h_m}^i(t)$ is the optimal control for problem (2.1) – (2.3), we have

$$S_{h_m}\left(\bar{\alpha}_{h_m}^i(t)\right) \le S_{h_m}\left(\alpha_m\left(x_{h_m}^i,t\right)\right) \tag{2.17}$$

Since the solution of problem (2.5) – (2.7) uniformly converges to the solution of the direct problem (2.1) – (2.3), the value of the approximating functional (2.8) converges to the value (2.4). Therefore, the following inequalities are also satisfied:

$$\left|S_{h_m}\left(\bar{\alpha}_{h_m}^i(t)\right) - S\left(\bar{\alpha}_{h_m}^i(x,t)\right)\right| < Ch, \tag{2.18}$$

$$\left|S_{h_m}\left(\alpha_m\left(x_{h_m}^i,t\right)\right) - S\left(\alpha_m(x,t)\right)\right| < Ch. \tag{2.19}$$

Considering equality

$$S\left(\alpha_{h_m}(x,t)\right) - S\big(\alpha_m(x,t)\big) =$$
$$= S\left(\alpha_{h_m}(x,t)\right) - S_{h_m}\left(\alpha^i_{h_m}(t)\right) + S_{h_m}\left(\bar{\alpha}^i_{h_m}(t)\right) - S\big(\alpha_m(x,t)\big)$$

the relation (2.17) – (2.19), we have

$$S\left(\alpha_{h_m}(x,t)\right) \leq S\big(\alpha_m(x,t)\big) + 2Ch \tag{2.20}$$

Since $\alpha_m(x,t)$ is a minimizing sequence, then from (2.20) it follows that the controls $\bar{\alpha}_{h_m}(x,t)$ is also a minimizing sequence of controls for (2.4) in problem (2.1) – (2.4), which is what should be proved.

Note that similar results are also valid in the case of the second boundary value problem, if within the boundaries of the segment $0 \leq x \leq 1$ values $u_x(x,t)$ are given. The presented scheme remains applicable even in the case when, under the boundary conditions (2.2), at one end of the segment $0 \leq x \leq 1$ values of $u(x,t)$are given, and at the other end, values of $u_x(x,t)$are given.

3. Results and discussion

As an example, let us consider the issues of determining the technological mode of operation of gas wells. Concerning dimensionless quantities, the problem can be formulated as follows: to control the bottomhole pressure $p_c(t)$ in a range specified based on technological considerations in such a way that the amount of gas extracted from wells deviates minimally from its planned value $q^*(t)$. As a measure of such deviation, a quadratic functional is taken

$$S = \frac{1}{2}\int_0^T [p_x^2(0,t) - q^*(t)]^2\, dt \tag{3.1}$$

where $p = p(x,t)$ describes the distribution of gas pressure in the "layer" $0 \leq x \leq 1$, which is a solution to the Leibenson equations [8]

$$p_t = 0.5 p_{xx}^2 \tag{3.2}$$

under the following initial and boundary conditions

$$p(x,0) = const = 1, 0 \leq x \leq 1, \tag{3.3}$$
$$p(0,t) = p_c(t), p_x(1,t) = 0, t > 0, \tag{3.4}$$

where conditions (3.3) and the first condition in (3.4) agree, that is, $p_c(0) = 1$.

Condition (3.3) means that at the initial moment, the formation was unperturbed with an initial constant pressure. The second condition in (3.4) indicates the impermeability of the boundary $x = 1$ of the formation. It is easy to see that equation (3.2) is obtained from equations (2.1) under $F(x,t,p,q,r,\alpha) \equiv pr + q.^2$

When approximating equation (3.2) by the method of straight lines, the problem was reduced to solving a variational problem associated with ordinary differential equations:

$$\frac{dp_h^i}{dt} = \frac{1}{2h^2}\left[\left(p_h^{i+1}\right)^2 - 2\left(p_h^i\right)^2 + \left(p_h^{i-1}\right)^2\right], 1 \leq i \leq N-1, \tag{3.5}$$
$$p_h^0 = p_c(t), p_h^N = p_h^{N-1}$$

with initial conditions

$$p_h^i(0) = 1{,}0 \leq i \leq N. \tag{3.6}$$

We need to select $p_c(t)$ so that the functional

$$S = \frac{1}{2h}\int_0^T [(p_h^1(t))^2 - (p_c(t))^2 - hq^*(t)]^2\, dt \qquad (3.7)$$

took a minimum value.

Introducing an additional variable $p_h^{N+1}(t)$, determined by the ratio

$$\frac{dp_h^{iN+1}}{dt} = \frac{1}{2h^2}[(p_h^1)^2 - (p_c)^2 - hq^*(t)], p_h^{N+1}(0) = 0, \qquad 3.8)$$

functional (3.7) can be represented as

$$S = p_h^{N+1}(T) \qquad (3.9)$$

Thus, problem (3.1) – (3.4) is replaced by the approximating system (3.5) – (3.6) and terminal functionals (3.9). A numerical solution to this problem using the gradient projection method was obtained in [6]. The sequence of controls $p_c^k(t),\ k = 0,1, \ldots$ found in this case with increasing number of iterations in a time interval $0 \le t \le T$ approached a given control, the optimality of which is known in advance. The value of the functional is equal to $\sim 4.8216 \cdot 10^{-6}$.

As a result of optimization, we obtained a solution

$$\max\left|\frac{\partial H}{\partial q}\right| \approx 0, \qquad (3.10)$$

which gives grounds to assume the existence of one locally optimal control, where the Hamilton function is approximating the optimal problem.

Thus, by regulating the bottomhole pressure $p_c(t)$ within certain limits, it is possible to maintain conditions in the bottom-hole zone area that determine the technological operating mode of gas wells.

4. Conclusion

The research conducted and numerical experiments allow us to draw the following conclusions:

1. Using the known estimate for solutions of a system of linear ordinary differential equations, it is proved that the solution of the approximating boundary value problem converges uniformly with speed $O(h^2)$ to the solution of the original boundary value problem.
2. It is established that the convergence of the method of straight lines in the functional in the approximating optimal problem takes place.
3. Depending on the selected class of admissible controls, a constructive scheme for building a minimizing sequence is proposed.

Authors' Declaration.
The authors declare that the work presented in this paper is original and has not been published elsewhere. All authors have contributed significantly to the research and writing of the manuscript and have approved the final version of the paper. The manuscript is not under consideration for publication elsewhere.

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Authors' Contribution Statement
Kamil Mamtiyev developed the theoretical framework, conducted the primary mathematical analysis, and formulated the main results.
Ulviyya Rzayeva contributed to the formulation of the optimal control problem, performed the convergence analysis, and coordinated the preparation of the manuscript.

Hafiz Bayramov was responsible for the literature review, numerical implementation, and validation of the proposed method.
Konul Mirzammadova contributed to the interpretation of results and assisted in editing and revising the final manuscript.

References

1. A. G. Butkovsky. Theory of Optimal Control of Systems with Distributed Parameters. Moscow: "Nauka", Main Editorial Board of Physical and Mathematical Literature. 1965, 474 p.
2. A. I. Egorov, L. N. Znamenskaya. On the Controllability of Oscillations of a System of Connected Objects with Distributed and Lumped Parameters, JVM and MF. 2006; v. 46, no. 6, pp. 1002–1018.
3. K. K. Mamtiyev, U. Sh. Rzayeva. Numerical solution of an optimal control problem for oscillations in coupled objects system. Eastern-European Journal of Enterprise Technologies. 2024, 2(2(128)), 64-74. doi: https: //doi. org/ 10.15587/1729-4061. 2024.301714.
4. M. Reyhanoglu, M. Jafari. Learning-Enabled Robust Control of Thermoacoustic Oscillations. Electronics. 2025; 14(9):1771. https://doi.org/10.3390/electronics14091771.
5. K. K. Mamtiyev, U. Ş. Rzayeva, A. Abdulova. Numerical solution of the control problem on the depletion of gas reservoirs with weakly permeable top. Eastern-European Journal of Enterprise Technologies. 2025; 2(6(122)), 15-23. doi: https: //doi. org/ 10.15587/1729-4061.2023.275986/.
6. A. Veliyev, E. Huseynov, K. Mamtiyev. Numerical Solition to One Problem of Optimal Control of Technological Operating Models of Gaz Wells to Prevent Pollu-tion, January 2025, DOI: 10.1007/978-3-031-81534-5_16in book international Conference on Smart Environment and Green Technologies-CSEGT2024 (pp.127-133).
7. R. Wu, Q. Jiang, B. Li, T. Liu, X. Zeng. Control Method for Ultra-Low Frequency Oscillation and Frequency Control Performance in Hydro–Wind Power Sending System. *Electronics*. 2024; 13(18):3691. https://doi.org/10.3390/electronics13183691.
8. P. Aboutalebi, F. M'zoughi, I. Martija, I. Garrido, A.J. Garrido. Switching Control Strategy for Oscillating Water Columns Based on Response Amplitude Operators for Floating Offshore Wind Turbines Stabilization. Applied Sciences. 2021; 11(11):5249. https://doi.org/10.3390/app11115249.
9. N. Kang. Low-Frequency Oscillations and Force Control Capabilities as a Function of Force Level in Older Women. Applied Sciences. 2022; 12(4):1812. https://doi.org/10.3390/app12041812.